

Bound states of four-body quantum problem of Wolfes type

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Abstract :

This work is devoted to the study of a particular four-body quantum problem of Wolfes type on the line [1]. We solve the corresponding Schrödinger equation in an harmonic confinement for the particles where we provide the full eigensolutions and corresponding eigenenergies. We discuss also the domain of the coupling constant where the irregular solution becomes physically acceptable. Finally, we extend the problem to D -dimensional space.

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1 Introduction

There exists a very limited number of exactly solvable many-body systems, even in one dimension (1D) [2, 3]. The Calogero model constitutes one of the famous ones which was exhaustively studied [4, 5]. A survey of many quantum integrable systems was done by Olshanetsky and Perelomov [6]. They classified the systems with respect to Lie algebras. Point interactions have also been considered, still in $D = 1$ dimensional space [7, 8].

The quest for exactly solvable non trivial quantum problems of few interacting particles on the line still retains attention. Early works of three-body linear problems of Calogero-Marchioro-Wolfes [9, 10, 11] have been followed by new extensions and cases. In a non exhaustive way, we can quote the three-body problems of Khare *et al.* [12], Quesne [13] and Meljanac *et al.* [14].

In a recent paper, we have exactly solved a generalization of the three-body Calogero-Marchioro-Wolfes problem with an additional non-translationally invariant three-body potential [15]. In this paper we propose a "completely solvable" model, i.e., a model where we give explicitly the eigenvalues and the complete set of eigensolutions of the problem. This completely solvable four-body bound states problem, in one and D -dimensional space, can be considered as a generalization of the four-body scattering linear problem proposed and solved by Wolfes [1] with additional non-translationally invariant four-body potentials.

The paper is organized as follows. In section 2 we expose and solve the problem for the harmonic case. The section 3 is devoted to extension to D -dimensional case. Our conclusions are drawn in section 4.

2 A generalization of the linear Wolfes four-body problem

We consider the four-body Hamiltonian on the line

$$H = \sum_{i=1}^4 \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + 2\lambda \sum_{i \neq j \neq k \neq m} \frac{1}{(x_i + x_j - x_k - x_m)^2} + \frac{4\mu}{(\sum_{i=1}^4 x_i)^2} + \frac{\beta}{\sum_{i=1}^4 x_i^2} \quad (1)$$

or more explicitly

$$\begin{aligned} H = & -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} + \omega^2(x_1^2 + x_2^2 + x_3^2 + x_4^2) + \frac{4\mu}{(x_1 + x_2 + x_3 + x_4)^2} \\ & + 4\lambda \left[\frac{1}{(x_1 + x_2 - x_3 - x_4)^2} + \frac{1}{(x_1 + x_3 - x_2 - x_4)^2} + \frac{1}{(x_1 + x_4 - x_2 - x_3)^2} \right] \\ & + \frac{\beta}{x_1^2 + x_2^2 + x_3^2 + x_4^2}. \end{aligned} \quad (2)$$

This Hamiltonian represents a system of four particles on the line with the same mass (with units $\hbar = 2m = 1$) interacting via only four-body potentials. One potential is translationally invariant with coupling constant λ of Wolfes type [1] and the two others with coupling constants μ, β are not translationally invariant. The whole system is confined in an harmonic oscillator trap.

The problem is solved in the following way. Setting

$$\begin{aligned} R &= \frac{x_1 + x_2 + x_3 + x_4}{2} \\ s &= \frac{x_1 + x_2 - x_3 - x_4}{2} \\ t &= \frac{x_1 + x_3 - x_2 - x_4}{2} \\ u &= \frac{x_1 + x_4 - x_2 - x_3}{2} \end{aligned} \quad (3)$$

the Hamiltonian, Eq.(2), becomes

$$\begin{aligned} H &= -\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \omega^2(R^2 + s^2 + t^2 + u^2) \\ &\quad + \frac{\mu}{R^2} + \lambda \left(\frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} \right) + \frac{\beta}{R^2 + s^2 + t^2 + u^2}. \end{aligned} \quad (4)$$

Note that, if $\beta = 0$, the problem is separable and the derivation of the solutions is straightforward. Otherwise, it is not separable in $\{R, s, t, u\}$ variables. To overcome this situation we introduce the following hyperspherical coordinates:

$$\begin{aligned} R &= r \cos \alpha, \quad s = r \sin \alpha \cos \theta, \quad t = r \sin \alpha \sin \theta \sin \varphi, \quad u = r \sin \alpha \sin \theta \cos \varphi, \\ 0 &\leq r < \infty, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi. \end{aligned} \quad (5)$$

The stationary Schrödinger equation is then written as:

$$\begin{aligned} &\left\{ -\frac{\partial^2}{\partial r^2} - \frac{3}{r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\beta}{r^2} + \frac{1}{r^2} \left[-\frac{\partial^2}{\partial \alpha^2} - 2 \cot \alpha \frac{\partial}{\partial \alpha} + \frac{\mu}{\cos^2 \alpha} \right. \right. \\ &+ \frac{1}{\sin^2 \alpha} \left(-\frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \frac{\lambda}{\cos^2 \theta} \right. \\ &\left. \left. \left. + \frac{1}{\sin^2 \theta} \left(-\frac{\partial^2}{\partial \varphi^2} + \frac{4\lambda}{\sin^2 2\varphi} \right) \right) \right] \right\} \Psi(r, \alpha, \theta, \varphi) = E \Psi(r, \alpha, \theta, \varphi), \end{aligned} \quad (6)$$

where $\Psi(r, \alpha, \theta, \varphi)$ represent the eigensolutions associated to eigenenergy E .

This four-body problem described by this equation (6) may be mapped to the problem of one particle in four dimensional space, with a non central potential of the form

$$V(r, \alpha, \theta, \varphi) = f_1(r) + \frac{1}{r^2} \left(f_2(\alpha) + \frac{1}{\sin^2 \alpha} \left[f_3(\theta) + \frac{f_4(\varphi)}{\sin^2 \theta} \right] \right). \quad (7)$$

It is then clear that the problem becomes separable in the four variables $\{r, \alpha, \theta, \varphi\}$. To find the solution we factorize the wave function as follows:

$$\Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) = \frac{F_{k,\ell,m,n}(r)}{r\sqrt{r}} \frac{G_{\ell,m,n}(\alpha)}{\sin \alpha} \frac{\Theta_{m,n}(\theta)}{\sqrt{\sin \theta}} \Phi_n(\varphi). \quad (8)$$

Accordingly, equation (6) separates in four decoupled differential equations:

$$\left(-\frac{d^2}{d\varphi^2} + \frac{4\lambda}{\sin^2 2\varphi}\right) \Phi_n(\varphi) = B_n \Phi_n(\varphi), \quad (9)$$

$$\left(-\frac{d^2}{d\theta^2} + \frac{(B_n - \frac{1}{4})}{\sin^2 \theta} + \frac{\lambda}{\cos^2 \theta}\right) \Theta_{m,n}(\theta) = C_{m,n} \Theta_{m,n}(\theta), \quad (10)$$

$$\left(-\frac{d^2}{d\alpha^2} + \frac{C_{m,n} - \frac{1}{4}}{\sin^2 \alpha} + \frac{\mu}{\cos^2 \alpha}\right) G_{\ell,m,n}(\alpha) = D_{\ell,m,n} G_{\ell,m,n}(\alpha), \quad (11)$$

and

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\beta + D_{\ell,m,n} - \frac{1}{4}}{r^2}\right) F_{k,\ell,m,n}(r) = E_{k,\ell,m,n} F_{k,\ell,m,n}(r). \quad (12)$$

In the interval $0 \leq \varphi \leq 2\pi$ the potential involved in equation (9) has a periodicity of π and has singularities at $\varphi = k\pi/2, k = 0, 1, 2, 3$. The equation is first solved in the interval $[0, \pi/2]$. It can be treated if and only if $\lambda > -1/4$ otherwise the operator has several self-adjoint extensions, each of them leading to a different spectrum [16, 17]. The regular eigensolutions of equation (9) on $[0, \pi/2]$ read [6, 18]

$$\Phi_n(\varphi) = (\sin 2\varphi)^{1/2+a} C_n^{(1/2+a)}(\cos 2\varphi) \quad (13)$$

where the $C_n^{(1/2+a)}$ denote the Gegenbauer Polynomials [19]. The corresponding eigenvalues are

$$B_n = 4 \left(\frac{1}{2} + a + n\right)^2 \quad (14)$$

with

$$n = 0, 1, 2, \dots \quad a = \frac{\sqrt{1+4\lambda}}{2} \quad (15)$$

The extension to the whole interval $[0, 2\pi]$ is made in two steps. First one, from $[0, \pi]$ up to $[\pi, 2\pi]$ using the periodicity of the solutions. On the other hand, the symmetric extension of the solutions in the interval $[\pi/2, \pi]$ reads

$$\Phi_n(\varphi) = (-\sin 2\varphi)^{1/2+a} C_n^{(1/2+a)}(\cos 2\varphi) \quad (16)$$

so that the real power of $(-\sin 2\varphi)$ is defined. In a compact form we obtain

$$\begin{aligned} \Phi_n(\varphi) &= (\epsilon_1 \sin 2\varphi)^{1/2+a} C_n^{(1/2+a)}(\cos 2\varphi) \\ \epsilon_1 &= \pm 1, \quad \frac{1-\epsilon_1}{2} \frac{\pi}{2} \leq \varphi \leq \frac{3-\epsilon_1}{2} \frac{\pi}{2} \end{aligned} \quad (17)$$

Note that when $a = 1/2, (\lambda = 0)$, a δ -pathology occurs at $\varphi = \pi/2$.

On the other hand, the antisymmetric extension of the solution reads

$$\begin{aligned} \Phi_n(\varphi) &= \text{sgn}(\sin(2\varphi)) (\epsilon_1 \sin 2\varphi)^{1/2+a} C_n^{(1/2+a)}(\cos 2\varphi) \\ \epsilon_1 &= \pm 1, \quad \frac{1-\epsilon_1}{2} \frac{\pi}{2} \leq \varphi \leq \frac{3-\epsilon_1}{2} \frac{\pi}{2} \end{aligned} \quad (18)$$

where $\text{sgn}(x) = x/|x|, x \neq 0$, denotes the sign of the variable x .

Generally, only the regular solution, $\Phi_n^{(+)}$, corresponding to $1/2 + a$, is retained. However, the irregular solution, $\Phi_n^{(-)}$, which is distinct from $\Phi_n^{(+)}$ for $\lambda > -1/4$, corresponding to $1/2 - a$ is physically acceptable when the Dirichlet condition is satisfied for $-1/4 < \lambda \leq 0$ (attractive potentials). If we release the Dirichlet condition, and ask only for the square integrability of the solution, as in [20], then $\Phi_n^{(-)}$ can be retained for $-1/4 < \lambda < 3/4$ [15]. For the irregular antisymmetric solution, a (derivative of) δ -pathology occurs at $\varphi = \pi/2$, when $a = 1/2$, ($\lambda = 0$).

Introducing the parameter

$$b_n = \sqrt{B_n} = 1 + 2a + 2n \quad (19)$$

(where we have only considered the positive root $b_n = \sqrt{B_n}$ because the other root $b_n = -\sqrt{B_n}$ leads to non-square integrable solutions for most values of n) the solution of Eq.(10) on the whole interval $[0, \pi]$ can be obtained as [6, 15]

$$\Theta_{m,n}(\theta) = \text{sgn}(\cos \theta)^{s_\theta} (\sin \theta)^{b_n+1/2} (\epsilon_2 \cos \theta)^{c+1/2} P_m^{(b_n, c)}(\cos 2\theta) \quad (20)$$

$$m = 0, 1, 2, \dots \quad c = \frac{1}{2} \sqrt{1 + 4\lambda} \quad (21)$$

$$\epsilon_2 = \pm 1, \quad \frac{1 - \epsilon_2}{2} \frac{\pi}{2} \leq \theta \leq \frac{3 - \epsilon_2}{2} \frac{\pi}{2}$$

corresponding to the eigenvalue

$$C_{m,n} = (2m + b_n + c + 1)^2. \quad (22)$$

In fact, $a = c$, if one takes the definitions Eqs.(15,21). In Eq.(20), the $P_m^{(b_n, c)}$ denote the Jacobi polynomials [19]. The value $s_\theta = 0, 1$ according to the fact that the solution has been extended in a symmetric (antisymmetric) way from $\theta \in [0, \pi/2]$ to $\theta \in [\pi/2, \pi]$. Note that, when $c = 1/2$, ($\lambda = 0$), a δ -pathology occurs for the symmetric solution at $\theta = \pi/2$.

The equation (11) is solved in the same manner as for equation (10). Setting

$$c_{m,n} = \sqrt{C_{m,n}} = 2m + b_n + c + 1 \quad (23)$$

(where we have only considered the positive root $c_{m,n} = \sqrt{C_{m,n}}$ because the other root $c_{m,n} = -\sqrt{C_{m,n}}$ leads to non-integrable solutions for most values of m) the solution writes as

$$G_{\ell, m, n}(\alpha) = \text{sgn}(\cos \alpha)^{s_\alpha} (\sin \alpha)^{c_{m,n}+1/2} (\epsilon_3 \cos \alpha)^{d+1/2} P_\ell^{(c_{m,n}, d)}(\cos 2\alpha) \quad (24)$$

$$\ell = 0, 1, 2, \dots \quad d = \frac{1}{2} \sqrt{1 + 4\mu} \quad (25)$$

$$\epsilon_3 = \pm 1 \quad \frac{1 - \epsilon_3}{2} \frac{\pi}{2} \leq \alpha \leq \frac{3 - \epsilon_3}{2} \frac{\pi}{2}$$

corresponding to the eigenvalue

$$D_{\ell, m, n} = (2\ell + c_{m,n} + d + 1)^2. \quad (26)$$

The value $s_\alpha = 0, 1$ according to the fact that the solution has been extended in a symmetric (antisymmetric) way from $\alpha \in [0, \pi/2]$ to $\alpha \in [\pi/2, \pi]$. Note that when $d = 1/2$, ($\mu = 0$), for symmetric solutions, a δ -pathology occurs at $\alpha = \pi/2$.

On the other hand, the reduced radial equation (12) is solved in the interval $0 \leq r < \infty$, with the condition of square integrability for the solutions. It implies $F_{k,\ell,m,n}(r) \rightarrow 0$ as $r \rightarrow \infty$.

We have to impose $\beta + D_{\ell,m,n} > 0$ in order to treat the centrifugal barrier in the vicinity of $r = 0$. Note that taking $\beta + D_{\ell,m,n} = 0$ leads to several self-adjoint extensions parametrized by a phase. This fact has been discussed in [15]. More details can be found in [21, 22]. Also attractive barriers $\beta + D_{\ell,m,n} < 0$, mentioned in [15], have been treated in [23, 24, 25]. Taking into account the definition of $D_{\ell,m,n}$, Eq.(26), we must have

$$\beta + D_{\ell,m,n} = \beta + (2\ell + 2m + 2n + 2a + c + d + 3)^2 > 0, \quad \forall n \geq 0, \quad \forall m \geq 0, \quad \forall \ell \geq 0. \quad (27)$$

For positive values a, c, d , the quantity $\beta + D_{\ell,m,n}$ is minimal for $n = 0, m = 0, \ell = 0$ and $a = c = d = 0$ (recall that $a \geq 0$, see (15), that $c \geq 0$, see (20) and that $d \geq 0$ see (25)). It puts constraint on β to satisfy $\beta > -9$ when a, c and d are positive. We introduce the auxiliary parameter $\kappa_{\ell,m,n}$ defined by

$$\kappa_{\ell,m,n}^2 = \beta + D_{\ell,m,n}, \quad \kappa_{\ell,m,n} = \sqrt{\beta + D_{\ell,m,n}}. \quad (28)$$

The solution of the radial equation (12) is [15]

$$F_{k,\ell,m,n}(r) = r^{\kappa_{\ell,m,n} + \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\kappa_{\ell,m,n})}(\omega r^2), \quad k = 0, 1, 2, \dots, \quad (29)$$

and it is associated to the eigenenergy

$$E_{k,\ell,m,n} = 2\omega(2k + \kappa_{\ell,m,n} + 1), \quad k = 0, 1, 2, \dots \quad (30)$$

The $L_k^{(q)}$ are the generalized Laguerre polynomials [19].

Taking into account all information, we conclude that the physically acceptable solutions of the Schrödinger equation (6) are

$$\begin{aligned} \Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) &= r^{\sqrt{\beta + (2\ell + 2m + 2n + 2a + c + d + 3)^2} - 1} e^{-\frac{\omega r^2}{2}} L_k^{(\sqrt{\beta + (2\ell + 2m + 2n + 2a + c + d + 3)^2})}(\omega r^2) \\ &\times \text{sgn}(\cos \alpha)^{s_\alpha} (\sin \alpha)^{2m + 2n + 2a + c + \frac{3}{2}} (\epsilon_3 \cos \alpha)^{d + 1/2} P_\ell^{(2m + 2n + 2a + c + 2, d)}(\cos 2\alpha) \\ &\times \text{sgn}(\cos \theta)^{s_\theta} (\sin \theta)^{2n + 2a + 1} (\epsilon_2 \cos \theta)^{c + \frac{1}{2}} P_m^{(2n + 2a + 1, c)}(\cos 2\theta) \\ &\times \text{sgn}(\sin(2\varphi))^{s_{2\varphi}} (\epsilon_1 \sin 2\varphi)^{a + \frac{1}{2}} C_n^{(a + \frac{1}{2})}(\cos 2\varphi), \\ k &= 0, 1, 2, \dots, \quad \ell = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots, \quad n = 0, 1, 2, \dots, \\ \frac{1 - \epsilon_1}{2} \frac{\pi}{2} &\leq \varphi \leq \frac{3 - \epsilon_1}{2} \frac{\pi}{2}, \quad \epsilon_1 = \pm 1, \\ \frac{1 - \epsilon_2}{2} \frac{\pi}{2} &\leq \theta \leq \frac{3 - \epsilon_2}{2} \frac{\pi}{2}, \quad \epsilon_2 = \pm 1, \quad \frac{1 - \epsilon_3}{2} \frac{\pi}{2} \leq \alpha \leq \frac{3 - \epsilon_3}{2} \frac{\pi}{2}, \quad \epsilon_3 = \pm 1 \\ a &= \frac{1}{2} \sqrt{1 + 4\lambda}, \quad c = \frac{1}{2} \sqrt{1 + 4\lambda}, \quad d = \frac{1}{2} \sqrt{1 + 4\mu}. \end{aligned} \quad (31)$$

Here $s_{2\varphi} = 0, 1$ according to the parity of the solution for $\varphi \in [0, \pi]$. We note that δ -pathologies occur for respectively $d = 1/2$, $a = c = 1/2$, in Eq.31, when symmetric solutions (i.e., s_α , s_θ or $s_{2\varphi}$ equal to zero) are considered.

The normalization constants $N_{k,\ell,m,n}$ can be calculated from

$$\begin{aligned} & \int_0^{+\infty} r^3 dr \int_{(1-\epsilon_3)\pi/4}^{(3-\epsilon_3)\pi/4} \sin^2(\alpha) d\alpha \int_{(1-\epsilon_2)\pi/4}^{(3-\epsilon_2)\pi/4} \sin \theta d\theta \int_{(1-\epsilon_1)\pi/4}^{(3-\epsilon_1)\pi/4} d\varphi \Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) \Psi_{k',\ell',m',n'}(r, \alpha, \theta, \varphi) \\ &= \delta_{k,k'} \delta_{\ell,\ell'} \delta_{m,m'} \delta_{n,n'} N_{k,\ell,m,n} \quad . \end{aligned} \quad (32)$$

Use is made, here, of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials [26].

The full expression of the eigenenergies is expressed by

$$\begin{aligned} E_{k,\ell,m,n} &\equiv E_{k,2\ell+2m+2n} = 2\omega \left(2k+1 + \sqrt{\beta + (2\ell+2m+2n+2a+c+d+3)^2} \right), \\ k &= 0, 1, 2, \dots, \quad \ell = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad n = 0, 1, 2, \dots, \end{aligned} \quad (33)$$

For illustration, the equation (31), multiplied by $r^{3/2}(\sin \alpha)\sqrt{\sin \theta}$ reads in Cartesian coordinates, for the symmetric case (we remind the reader that $a = c = \sqrt{1/4 + \lambda}$)

$$\begin{aligned} \Psi(x_1, x_2, x_3, x_4) &\propto |(x_1 + x_4 - x_2 - x_3)(x_1 + x_3 - x_2 - x_4)(x_1 + x_2 - x_3 - x_4)|^{a+1/2} \\ &\times [(x_1 - x_2)^2 + (x_3 - x_4)^2]^n C_n^{a+1/2} \left(\frac{2(x_1 - x_2)(x_4 - x_3)}{(x_1 - x_2)^2 + (x_3 - x_4)^2} \right) \\ &\times [4(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2]^m \\ &\times P_m^{(2n+2a+1,a)} \left(\frac{8(x_1 x_2 + x_3 x_4) - (x_1 + x_2 + x_3 + x_4)^2}{4(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2} \right) \\ &\times |x_1 + x_2 + x_3 + x_4|^{d+1/2} (x_1^2 + x_2^2 + x_3^2 + x_4^2)^q \\ &\times P_\ell^{(2m+2n+3a+2,d)} \left(\frac{2(x_1 + x_2 + x_3 + x_4)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}{x_1^2 + x_2^2 + x_3^2 + x_4^2} \right) \\ &\times \exp(-\omega(x_1^2 + x_2^2 + x_3^2 + x_4^2)/2) L_k^p(\omega(x_1^2 + x_2^2 + x_3^2 + x_4^2)) \end{aligned} \quad (34)$$

with

$$\begin{aligned} p &= \sqrt{\beta + (2\ell + 2m + 2n + 3a + d + 3)^2} \\ q &= p/2 - (2m + 2n + 3a + d + 3)/2 . \end{aligned}$$

The eigenvalue Eq.(33) shows a degeneracy, i.e., all solutions such that the equality $\ell + m + n = N$ is satisfied, N being fixed, correspond to the same eigenvalue. This imply that any combination of solutions such that $\ell + m + n$ is constant is also solution. We can then obtain, $k, \ell, m + n$ being fixed, a unique solution, symmetric in the permutation on the set of variables $\{x_1, x_2, x_3, x_4\}$ when $n + m$ is even. This latter symmetric solution is in fact proportional to $\sum_{\sigma \in S_4} \Psi(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ where S_4 denotes the permutation group of four elements.

Let us now consider the irregular "polynomial" solutions corresponding to $1/2 - a$. We have to replace a by $-a$ in all equations, from Eq.(17) until Eq.(33). Recall that for $-1/4 < \lambda < 3/4$, the irregular solutions, Eq.(17), are square integrable, as seen before. The Sturm-Liouville operator (9) is self-adjoint for $\lambda \neq -1/4$. It has to be added that, a δ pathology occurs in (31) for $a = 1/2$ ($\lambda = 0$). We consider values of λ in $] -1/4, 0[\cup]0, 3/4[$.

We next examine the impact on the change $a \mapsto -a$ on the function $\Theta_{m,n}(\theta)$, Eq.(20). We remind the reader that $|c| = |a|$, $a = \pm\sqrt{1/4 + \lambda}$. Here $a = -\sqrt{1/4 + \lambda}$. Also, we allow $c = \pm\sqrt{1/4 + \lambda}$ i.e., we consider negative values of c . We have $|a| = |c|$ but we allow $a = \pm c$ (both quantities being allowed to be negative and having a different sign). Note that another situation corresponds to $a = \sqrt{1/4 + \lambda} = -c$. But now we concentrate on $a = -\sqrt{1/4 + \lambda}$, $c = \pm a$.

First, to ensure the self-adjointness of the Sturm-Liouville operator Eq.(10), we have to impose both $B_n \neq 0$ for every n and $\lambda \neq -1/4$. Taking into account Eq.(19), for a changed into $-a$, we have clearly to ask for $a \neq 1/2$ which is equivalent to $\lambda \neq 0$. These latter conditions are fulfilled for λ in $] -1/4, 0[\cup]0, 3/4[$.

Then we examine for which value of a we obtain square integrable solutions $\Theta_{m,n}(\theta)$ for every value of n . Since

$$\frac{1}{2} + b_n = 2n + \frac{3}{2} - 2a \quad (a = \frac{1}{2}\sqrt{1 + 4\lambda}) , \quad (35)$$

we have

$$(\forall n \geq 0) \quad \frac{1}{2} + b_n \geq \frac{3}{2} - 2a . \quad (36)$$

The function $\Theta_{m,n}(\theta)$, Eq.(20), leads to square integrable solutions for every value of n if $a < 1$, corresponding to $b_n + 1 > 0$, $(\forall n)$. This latter inequality happens for $\lambda < 3/4$. This is satisfied when $\lambda \in] -1/4, 0[\cup]0, 3/4[$ (see the paragraph below Eq.(17)). As far as the term $\lambda/\cos^2 \theta$ is concerned irregular solutions happen when $c = \sqrt{1/4 + \lambda}$ is changed in $-c$ in Eq.(20). These solutions are square integrable when $c < 1$ i.e., for $\lambda < 3/4$.

Consider the last angular equation concerning the variable α . The differential operator is self-adjoint provided that

$$(\forall m \geq 0)(\forall n \geq 0) \quad c_{m,n} = 2m + 2n + 2 \pm c - 2a \geq 2 - 2a \pm c , \quad (37)$$

is non zero. So that the self-adjointness is ensured when $2 - 2a \pm c > 0$ or equivalently $2 - 2a \pm a > 0$. The quantity $2 - a$ is always positive in the domain of acceptable λ whereas $2 - 3a$ is positive for $\lambda < 7/36 \simeq 0.194444$. Also μ has to be different from $-1/4$.

The square integrability of the function $G_{\ell,m,n}$, Eq.(24), is ensured by $c_{m,n} + 1 > 0$, $(\forall m)$ $(\forall n)$ i.e. when $3 - 2a \pm c > 0$ or equivalently $3 - 2a \pm a > 0$. Both quantities $3 - 2a \pm a$ are always positive for $-1/4 < \lambda < 3/4$. This defines a domain in λ of acceptable solutions. Also if d is allowed to be negative we must have $-1/4 < d < 3/4$.

As far as the radial equation is concerned, the constraint $\beta + D_{\ell,m,n} > 0$ allows us to treat the centrifugal barrier in the vicinity of $r = 0$. (see the above discussion for $\beta + D_{\ell,m,n} \leq 0$.) Taking into account the definition of $D_{\ell,m,n}$, Eq.(26), we have

$$\beta + D_{\ell,m,n} = \beta + (2\ell + 2m + 2n - 2a \pm c + d + 3)^2 > 0, \quad \forall n \geq 0, \quad \forall m \geq 0, \quad \forall \ell \geq 0. \quad (38)$$

This is satisfied for every $\{\ell, m, n\}$ when $\beta > 0$ and (see Eq.(38))

$$\pm c - 2a + 3 + d > \sqrt{-\beta} \quad \beta \leq 0 \quad (39)$$

This condition defines a domain of acceptable values of β depending on the values of λ reminding that $\lambda \in] -1/4, 0[\cup]0, 3/4[$. Under such conditions, the radial solutions, Eq.(29), are square integrable because $\kappa_{\ell,m,n} > 0$.

Note that the spectrum for irregular solutions has eigenvalues lower than the ones corresponding to the regular solutions. This spectrum is given by

$$\begin{aligned} E_{k,\ell,m,n}^{(<)} &\equiv E_{k,\ell+m+n}^{(<)} = 2\omega \left(2k+1 + \sqrt{\beta + (2\ell + 2m + 2n \pm c + d - 2a + 3)^2} \right), \\ k &= 0, 1, 2, \dots, \quad \ell = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad n = 0, 1, 2, \dots, . \end{aligned} \quad (40)$$

Also d can be changed in $-d = -\sqrt{1/4 + \mu}$. The requirement

$$\begin{aligned} \pm c - 2a + 3 - d &> \sqrt{-\beta} & \beta &\leq 0 \\ \pm c - 2a + 3 - d &> 0 & \beta &\geq 0 \end{aligned} \quad (41)$$

ensures the self-adjointness of Eq.(12).

3 D-dimension

The generalization to the D -dimensional space follows the same strategy as in section 2. We consider the Hamiltonian:

$$H = \sum_{i=1}^4 \left(-\Delta_i + \omega^2 \vec{r}_i^2 \right) + 2\lambda \sum_{i \neq j \neq k \neq m} \frac{1}{(\vec{r}_i + \vec{r}_j - \vec{r}_k - \vec{r}_m)^2} + \frac{4\mu}{(\sum_{i=1}^4 \vec{r}_i)^2} + \frac{\beta}{\sum_{i=1}^4 \vec{r}_i^2} \quad (42)$$

Setting as in section 2

$$\begin{aligned} \vec{R} &= \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4}{2} \\ \vec{s} &= \frac{\vec{r}_1 + \vec{r}_2 - \vec{r}_3 - \vec{r}_4}{2} \\ \vec{t} &= \frac{\vec{r}_1 + \vec{r}_3 - \vec{r}_2 - \vec{r}_4}{2} \\ \vec{u} &= \frac{\vec{r}_1 + \vec{r}_4 - \vec{r}_2 - \vec{r}_3}{2} \end{aligned} \quad (43)$$

the Hamiltonian Eq.(42) becomes

$$\begin{aligned} H &= -\Delta_{\vec{R}} - \Delta_{\vec{s}} - \Delta_{\vec{t}} - \Delta_{\vec{u}} + \omega^2(R^2 + s^2 + t^2 + u^2) \\ &\quad + \frac{\mu}{R^2} + \lambda \left(\frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2} \right) + \frac{\beta}{R^2 + s^2 + t^2 + u^2}. \end{aligned} \quad (44)$$

Since the potential does not depend on the angles between $\vec{R}, \vec{s}, \vec{t}, \vec{u}$, the separation of angular and radial variables, together with the use of the hyperspherical harmonics [19], allows us to write

$$\Phi(\vec{R}, \vec{s}, \vec{t}, \vec{u}) = \frac{\Phi^{(\ell_r, \ell_s, \ell_t, \ell_u)}(R, s, t, u)}{(Rstu)^{(D-1)/2}} Y_{\ell_R, [M_R]}(\Omega_R) Y_{\ell_s, [M_s]}(\Omega_s) Y_{\ell_t, [M_t]}(\Omega_t) Y_{\ell_u, [M_u]}(\Omega_u) . \quad (45)$$

Here $[M]$ denotes the set $[M] = \{m_1, m_2, \dots, m_p\}$, $p = D - 2$ satisfying $\ell = m_0 \geq m_1 \geq m_2 \dots \geq m_p \geq 0$ and [19]

$$Y_{\ell, [M]} = e^{\pm i m_p \phi} \prod_{k=1}^p (\sin \theta_k)^{m_k} \prod_{k=0}^{p-1} C_{m_k - m_{k+1}}^{m_{k+1} + p/2 - k/2}(\cos \theta_{k+1}) \quad (46)$$

where the hyperspherical harmonics $Y_{\ell, [M]}$ are given in terms of the Gegenbauer polynomials $C_n^a(x)$. For $D = 2$, $Y_\ell = \exp(i\ell\phi)$, $\ell \in \mathbb{Z}$ and $\phi \in [0, 2\pi]$. Recall that the hyperspherical polar coordinates are given by

$$\begin{aligned} x_1 &= r \cos \theta_1 \\ x_2 &= r \sin \theta_1 \cos \theta_2 \\ x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ &\dots \\ x_{p+1} &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_p \cos \phi \\ x_{p+2} &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_p \sin \phi \end{aligned} \quad (47)$$

with $\theta_k, \in [0, \pi]$, $k = 1, 2, \dots, p$ and $\phi \in [0, 2\pi]$. Setting $(D - 3)/2 = md$ (here $md \geq -1/2$) we obtain

$$\begin{aligned} &\left(-\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \omega^2(R^2 + s^2 + t^2 + u^2) + \frac{\mu + (\ell_R + md)(\ell_R + md + 1)}{R^2} \right. \\ &+ \frac{\lambda + (\ell_s + md)(\ell_s + md + 1)}{s^2} + \frac{\lambda + (\ell_t + md)(\ell_t + md + 1)}{t^2} \\ &\left. + \frac{\lambda + (\ell_u + md)(\ell_u + md + 1)}{u^2} + \frac{\beta}{R^2 + s^2 + t^2 + u^2} \right) \Phi^{(\ell_r, \ell_s, \ell_t, \ell_u)}(R, s, t, u) = 0 \end{aligned} \quad (48)$$

We introduce the following hyperspherical coordinates:

$$\begin{aligned} R &= r \cos \alpha, \quad s = r \sin \alpha \cos \theta, \quad t = r \sin \alpha \sin \theta \sin \varphi, \quad u = r \sin \alpha \sin \theta \cos \varphi, \\ 0 &\leq r < \infty, \quad 0 \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq \frac{\pi}{2}. \end{aligned} \quad (49)$$

Here $\alpha, \theta, \varphi \in [0, \pi/2]$ because R, s, t, u are positive.

The stationary Schrödinger equation is then written:

$$\begin{aligned}
& \left\{ -\frac{\partial^2}{\partial r^2} - \frac{3}{r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\beta}{r^2} + \frac{1}{r^2} \left[-\frac{\partial^2}{\partial \alpha^2} - 2 \cot \alpha \frac{\partial}{\partial \alpha} + \frac{\mu + (\ell_R + md)(\ell_R + md + 1)}{\cos^2 \alpha} \right. \right. \\
& + \frac{1}{\sin^2 \alpha} \left(-\frac{\partial^2}{\partial \theta^2} - \cot \theta \frac{\partial}{\partial \theta} + \frac{\lambda + (\ell_s + md)(\ell_s + md + 1)}{\cos^2 \theta} \right. \\
& + \frac{1}{\sin^2 \theta} \left(-\frac{\partial^2}{\partial \varphi^2} + \frac{\lambda + (\ell_t + md)(\ell_t + md + 1)}{\sin^2 \varphi} \right. \\
& \left. \left. \left. + \frac{\lambda + (\ell_u + md)(\ell_u + md + 1)}{\cos^2 \varphi} \right) \right] \right\} \Psi(r, \alpha, \theta, \varphi) = E \Psi(r, \alpha, \theta, \varphi), \tag{50}
\end{aligned}$$

where $\Psi(r, \alpha, \theta, \varphi)$ represents the eigensolution associated to eigenenergy E .

Assuming

$$\Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) = \frac{F_{k,\ell,m,n}(r)}{r\sqrt{r}} \frac{G_{\ell,m,n}(\alpha)}{\sin \alpha} \frac{\Theta_{m,n}(\theta)}{\sqrt{\sin \theta}} \Phi_n(\varphi) \tag{51}$$

we obtain

$$\left(-\frac{d^2}{d\varphi^2} + \frac{\lambda + (\ell_t + md)(\ell_t + md + 1)}{\sin^2 \varphi} + \frac{\lambda + (\ell_u + md)(\ell_u + md + 1)}{\cos^2 \varphi} \right) \Phi_n(\varphi) = B_n \Phi_n(\varphi), \tag{52}$$

$$\left(-\frac{d^2}{d\theta^2} + \frac{(B_n - \frac{1}{4})}{\sin^2 \theta} + \frac{\lambda + (\ell_s + md)(\ell_s + md + 1)}{\cos^2 \theta} \right) \Theta_{m,n}(\theta) = C_{m,n} \Theta_{m,n}(\theta), \tag{53}$$

$$\left(-\frac{d^2}{d\alpha^2} + \frac{C_{m,n} - \frac{1}{4}}{\sin^2 \alpha} + \frac{\mu + (\ell_R + md)(\ell_R + md + 1)}{\cos^2 \alpha} \right) G_{\ell,m,n}(\alpha) = D_{\ell,m,n} G_{\ell,m,n}(\alpha), \tag{54}$$

and

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\beta + D_{\ell,m,n} - \frac{1}{4}}{r^2} \right) F_{k,\ell,m,n}(r) = E_{k,\ell,m,n} F_{k,\ell,m,n}(r). \tag{55}$$

The equation (52) is solved in the interval $[0, \pi/2]$. The regular eigensolutions of equation (52) on $[0, \pi/2]$ read [6, 18]

$$\Phi_n(\varphi) = (\sin \varphi)^{1/2+a} (\cos \varphi)^{1/2+b} P_n^{(a,b)}(\cos 2\varphi) \tag{56}$$

where the $P_n^{(a,b)}$ denote the Jacobi Polynomials [19]. The corresponding eigenvalues are

$$B_n = (a + b + 1 + 2n)^2 \tag{57}$$

with

$$n = 0, 1, 2, \dots \quad a = \sqrt{\lambda + (\ell_t + md + 1/2)^2} \quad b = \sqrt{\lambda + (\ell_u + md + 1/2)^2} \tag{58}$$

The operator in Eq.(52) is self-adjoint for every ℓ_t, ℓ_u provided that $\lambda + (md + 1/2)^2 > 0$. Introducing the parameter

$$b_n = \sqrt{B_n} = 1 + a + b + 2n \tag{59}$$

(where we have only considered the positive root $b_n = \sqrt{B_n}$ because the other root $b_n = -\sqrt{B_n}$ leads to non-square integrable solutions for most values of n) the solution of Eq.(53) on the interval $[0, \pi/2]$

can be obtained as [6, 15]

$$\begin{aligned} \Theta_{m,n}(\theta) &= (\sin \theta)^{b_n+1/2} (\cos \theta)^{c+1/2} P_m^{(b_n,c)}(\cos 2\theta) \\ m &= 0, 1, 2, \dots \quad c = \sqrt{\lambda + (\ell_s + md + 1/2)^2} \end{aligned} \quad (60)$$

corresponding to the eigenvalue

$$C_{m,n} = (2m + b_n + c + 1)^2. \quad (61)$$

In Eq.(60) the $P_m^{(b_n,c)}$ denote the Jacobi polynomials [19]. The operator in Eq.(53) is self-adjoint for every ℓ_t, ℓ_u, ℓ_s provided that $\lambda + (md + 1/2)^2 > 0$. The second condition namely $B_n \neq 0$ i.e. $2\sqrt{\lambda + (md + 1/2)^2} + 1 > 0$ is always satisfied.

The equation (54) is solved in the same manner as done for equation (53). Setting

$$c_{m,n} = \sqrt{C_{m,n}} = 2m + b_n + c + 1 \quad (62)$$

(where we have only considered the positive root $c_{m,n} = \sqrt{C_{m,n}}$ because the other root $c_{m,n} = -\sqrt{C_{m,n}}$ leads to non-integrable solutions for most values of m) the solution writes as

$$G_{\ell,m,n}(\alpha) = (\sin \alpha)^{c_{m,n}+1/2} (\cos \alpha)^{d+1/2} P_\ell^{(c_{m,n},d)}(\cos 2\alpha) \quad (63)$$

$$\ell = 0, 1, 2, \dots \quad d = \sqrt{\mu + (\ell_R + md + 1/2)^2} \quad (64)$$

corresponding to the eigenvalue

$$D_{\ell,m,n} = (2\ell + c_{m,n} + d + 1)^2. \quad (65)$$

The operator in Eq.(54) is self-adjoint for every $\ell_t, \ell_u, \ell_s, \ell_R$ provided that $\lambda + (md + 1/2)^2 > 0$. The condition $C_{m,n} \neq 0$ i.e. $3\sqrt{\lambda + (md + 1/2)^2} + 2 > 0$ is always satisfied.

On the other hand, the reduced radial equation (55) is solved in the interval $0 \leq r < \infty$, with the condition of square integrability for the solutions. It implies $F_{k,\ell,m,n}(r) \rightarrow 0$ as $r \rightarrow \infty$. We have to impose $\beta + D_{\ell,m,n} > 0$ in order to treat the centrifugal barrier in the vicinity of $r = 0$.

$$\beta + D_{\ell,m,n} = \beta + (2\ell + 2m + 2n + a + b + c + d + 3)^2 > 0, \quad \forall n \geq 0, \quad \forall m \geq 0, \quad \forall \ell \geq 0. \quad (66)$$

For positive a, b, c, d the quantity $\beta + D_{\ell,m,n}$ is minimal for $n = 0, m = 0, \ell = 0$ and $a = b = c = d = 0$ (recall that $a, b \geq 0$, see (58), that $c \geq 0$, see (60) and that $d \geq 0$ see (64)) It put constraint on β to satisfy $\beta > -9$ when a, b, c and d are positive. We introduce the auxiliary parameter $\kappa_{\ell,m,n}$ defined by

$$\kappa_{\ell,m,n}^2 = \beta + D_{\ell,m,n}, \quad \kappa_{\ell,m,n} = \sqrt{\beta + D_{\ell,m,n}}. \quad (67)$$

The solution of the radial equation (55) is [15]

$$F_{k,\ell,m,n}(r) = r^{\kappa_{\ell,m,n} + \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\kappa_{\ell,m,n})}(\omega r^2), \quad k = 0, 1, 2, \dots, \quad (68)$$

and it is associated to the eigenenergy

$$E_{k,\ell,m,n} = 2\omega(2k + \kappa_{\ell,m,n} + 1), \quad k = 0, 1, 2, \dots \quad (69)$$

The $L_k^{(q)}$ are the generalized Laguerre polynomials [19].

Taking into account all information, we conclude that the physically acceptable solutions of the Schrödinger equation (50) are

$$\begin{aligned}
\Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) &= r^{\sqrt{\beta+(2\ell+2m+2n+a+b+c+d+3)^2}-1} e^{-\frac{\omega r^2}{2}} L_k^{\left(\sqrt{\beta+(2\ell+2m+2n+a+b+c+d+3)^2}\right)}(\omega r^2) \\
&\times (\sin \alpha)^{2m+2n+a+b+c+\frac{5}{2}} (\cos \alpha)^{d+1/2} P_\ell^{(2m+2n+a+b+c+2,d)}(\cos 2\alpha) \\
&\times (\sin \theta)^{2n+a+b+\frac{3}{2}} (\cos \theta)^{c+\frac{1}{2}} P_m^{(2n+a+b+1,c)}(\cos 2\theta) \\
&\times (\sin \varphi)^{a+\frac{1}{2}} (\cos \varphi)^{b+\frac{1}{2}} P_n^{(a,b)}(\cos 2\varphi), \\
k &= 0, 1, 2, \dots, \quad \ell = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{70}$$

The normalization constants $N_{k,\ell,m,n}$ can be calculated from

$$\begin{aligned}
&\int_0^{+\infty} r^3 dr \int_0^{\pi/2} \sin^2(\alpha) d\alpha \int_0^{\pi/2} \sin \theta d\theta \int_0^{\pi/2} d\varphi \Psi_{k,\ell,m,n}(r, \alpha, \theta, \varphi) \Psi_{k',\ell',m',n'}(r, \alpha, \theta, \varphi) \\
&= \delta_{k,k'} \delta_{\ell,\ell'} \delta_{m,m'} \delta_{n,n'} N_{k,\ell,m,n},
\end{aligned} \tag{71}$$

use is made, here, of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials [26].

The full expression of the eigenenergies is expressed by

$$\begin{aligned}
E_{k,\ell,m,n} &\equiv E_{k,\ell+m+n} = 2\omega \left(2k + 1 + \sqrt{\beta + (2\ell + 2m + 2n + a + b + c + d + 3)^2} \right), \\
k &= 0, 1, 2, \dots, \quad \ell = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad n = 0, 1, 2, \dots, .
\end{aligned} \tag{72}$$

4 Conclusions

In this paper, we have proposed and solved a linear four-body quantum problem of Wolfes type, in the case of bound states. The operator we consider is composed of an Harmonic trap with four-body translationally invariant potential proposed by Wolfes with additional non-translationally invariant four-body potential. We explicitly give the full solutions of the corresponding Schrödinger equation, namely the wave functions in terms of the angular and radial variables together with the energy spectrum.

We have been able, also, to generalize and solve this linear problem to any D -dimensional space where the explicit eigensolutions and the corresponding spectrum have been calculated.

We also investigate the domain of coupling constant where the irregular solutions become square integrable and physically acceptable.

This proposed exactly solved four-body problem can be generalized to a "Coulomb-type" problem where the Harmonic trap can be replaced by an attractive "Coulomb" type potential as done in [12] for the three-body problem.

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